

Cusps of Hilbert modular varieties

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Abstract

Motivated by a question of Hirzebruch on the possible topological types of cusp cross-sections of Hilbert modular varieties, we give a necessary and sufficient condition for a manifold M to be diffeomorphic to a cusp cross-section of a Hilbert modular variety. Specialized to Hilbert modular surfaces, this proves that every Sol 3-manifold is diffeomorphic to a cusp cross-section of a (generalized) Hilbert modular surface. We also deduce an obstruction to geometric bounding in this setting. Consequently, there exist Sol 3-manifolds that cannot arise as a cusp cross-section of a 1-cusped nonsingular Hilbert modular surface.

1 Introduction

Main results

It is a classical problem in topology to decide whether or not a closed n -manifold M bounds. Hamrick and Royster [5] resolved this in the affirmative for flat n -manifolds and Rohlin [12] for closed 3-manifolds. However, beyond these two classes there are few other settings where the story is nearly this complete. The introduction of geometry to a topological problem provides additional structure which can lead to new insight. This philosophy serves as motivation for the primary concern of this article which is a geometric notion of bounding and its specialization to infrasolv manifolds.

Let k be a totally real number field with $[k : \mathbf{Q}] = n$, \mathcal{O}_k the ring of integers of k , and $\sigma_1, \dots, \sigma_n$ denote the n real embeddings of k . The group $\mathrm{PSL}(2; \mathcal{O}_k)$ is an arithmetic subgroup of the n -fold product $(\mathrm{PSL}(2; \mathbf{R}))^n$ (see [2]) via the embedding $\xi \mapsto (\sigma_1(\xi), \dots, \sigma_n(\xi))$ for $\xi \in \mathrm{PSL}(2; \mathcal{O}_k)$. Through this embedding, $\mathrm{PSL}(2; \mathcal{O}_k)$ acts with finite volume on the n -fold product of real hyperbolic planes $(\mathbf{H}^2)^n$. The group $\mathrm{PSL}(2; \mathcal{O}_k)$ is called *the Hilbert modular group*. More generally, we call any subgroup Λ of $\mathrm{PSL}(2; k)$ which is commensurable with $\mathrm{PSL}(2; \mathcal{O}_k)$ a *Hilbert modular group* and the quotients $(\mathbf{H}^2)^n / \Lambda$, *Hilbert modular varieties*. In the case that k is a real quadratic number field, these quotients are called *Hilbert modular surfaces*. For more on Hilbert modular surfaces, see [6] or [16].

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The primary focus of this article is cusp cross-sections of Hilbert modular varieties. These infrasolv manifolds are virtual n -torus bundles over $(n-1)$ -tori where $[k : \mathbf{Q}] = n$ and $\text{rank } \mathcal{O}_k^\times = n-1$. For brevity, we simply call these *virtual $(n, n-1)$ -torus bundles*. Recall that an n -torus bundle over an m -torus is the total space of a fiber bundle with base manifold T^m and fiber T^n . We call such manifolds simply *(n, m) -torus bundles*. We say that N is a *virtual (n, m) -torus bundle* if N is finitely covered by an (n, m) -torus bundle.

In [9], cusp cross-sections of real, complex, and quaternionic arithmetic hyperbolic n -orbifolds were classified. In this article, we continue this theme by classifying cusp cross-sections of Hilbert modular varieties. By taking the quotient of the associated neutered space for the Hilbert modular group Λ , we obtain a compact Riemannian $2n$ -orbifold whose totally geodesic boundaries are the cusp cross-sections equipped with metrics (defined up to scaling) coming from the associated solvable Lie group.

Before stating our first classification result, we introduce an additional piece of terminology.

For a totally real number field k , we say $\beta \in k$ is *totally positive* if $\sigma_j(\beta) > 0$ for $j = 1, \dots, n$. We denote the set of totally positive elements and totally positive integers by k_+ and $\mathcal{O}_{k,+}$, and define the sets $k_+^\times = k_+ \cap k^\times$, $\mathcal{O}_{k,+}^\times = \mathcal{O}_k^\times \cap \mathcal{O}_{k,+}$. We say that a virtual torus bundle N is *k -defined* if there exists a faithful representation $\rho : \pi_1(N) \rightarrow k \rtimes k_+^\times$. If in addition $\rho(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$, we say that N is *k -arithmetic*.

Our first result is:

Theorem 1.1. *A virtual $(n, n-1)$ -torus bundle N is diffeomorphic to a cusp cross-section of a Hilbert modular variety over k if and only if $\pi_1(N)$ is k -arithmetic.*

Theorem 1.1 answers a question of Hirzebruch [6, page 203] who asked (in our terminology) which k -arithmetic torus bundles arise as cusp cross-sections of Hilbert modular varieties. See Subsection 3.3 for more on this.

Every $(2, 1)$ -torus bundle admits either a Euclidean, Nil, or Sol geometry. Long and Reid [8] proved that the $(2, 1)$ -torus bundles which admit a Euclidean structure are diffeomorphic to cusp cross-sections of arithmetic real hyperbolic 4-orbifolds. In [9], we proved that those that admit Nil structures are diffeomorphic to cusp cross-sections of arithmetic complex hyperbolic 2-orbifolds. In this article, we prove (see §5 for the definitions):

Theorem 1.2. *Every Sol 3-manifold is diffeomorphic to a cusp cross-section of a generalized Hilbert modular surface.*

We note that this shows closed 3-manifolds modelled on these three geometries bound; of course, this is not new as Rohlin proved this for any 3-manifold.

Using the Atiyah-Patodi-Singer signature formula, Long and Reid [7] showed that a flat 3-manifold which arises as a cusp cross-section of a 1-cusped real hyperbolic 4-manifold must have integral η -invariant. Together with Ouyang's work, this proves

that certain flat 3-manifolds cannot be the cusp cross-section of a 1-cusped real hyperbolic 4-manifold. We conclude this article with a similar result. Specifically, using the work of Hirzebruch [6], Atiyah-Donnelly-Singer [1], and Cheeger-Gromov [3], we prove:

Theorem 1.3. *There exists a Sol 3-manifold which cannot be diffeomorphic to a cusp cross-section of any 1-cusped Hilbert modular surface with torsion free fundamental group.*

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2 Preliminary material

2.1 Stabilizer groups

For $v \in \partial \mathbf{H}^n$, the group $\text{Stab}(v) = \{\gamma \in \text{Isom}(\mathbf{H}^n) : \gamma v = v\}$ is isomorphic to $\mathbf{R}^{n-1} \rtimes (\mathbf{R}^+ \times \text{O}(n-1))$. For $v \in \partial \mathbf{H}^n$ and $H < \text{Isom}(\mathbf{H}^n)$, we define the *stabilizer group of H at v* to be $\Delta_v(H) = \text{Stab}(v) \cap H$. When $\Delta_v(H)$ contains a parabolic isometry, we call $\Delta_v(H)$ the *maximal peripheral subgroup of H at v* and say that H has a *cuspidal* at v . Often, we simply write $\Delta(H)$.

Cusps, horospheres, and cusp cross-sections are defined as in the hyperbolic setting via Iwasawa decompositions of $(\text{PSL}(2; \mathbf{R}))^r$. For the Hilbert modular group $\text{PSL}(2; \mathcal{O}_k)$ over a totally real number field k , the stabilizer of the boundary point corresponding to the Iwasawa decomposition given by the r -fold product of the groups $\mathbf{A}, \mathbf{N}, \mathbf{K}$ is the peripheral subgroup

$$\Delta = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} : \alpha \in \mathcal{O}_k, \beta \in \mathcal{O}_{k,+}^\times \right\}.$$

Every peripheral subgroup of $\text{PSL}(2; \mathcal{O}_k)$ is conjugate in $\text{PSL}(2; k)$ to a group commensurable with Δ .

2.2 Infrasolv manifolds and smooth rigidity

For a simply connected, connected solvable Lie group S , the affine group of S is $\text{Aff}(S) = S \rtimes \text{Aut}(S)$. We say that a discrete subgroup $\Gamma < \text{Aff}(S)$ is an *infrasolv group modelled on S* if $\Gamma \cap S$ is finite index in Γ and S/Γ is compact. An infrasolv group which is a subgroup of S will be called a *solv group modelled on S* . Any smooth manifold which is diffeomorphic to S/Γ for some infrasolv group will be called an *infrasolv manifold modelled on S* . When Γ is a solv group, we call the manifold S/Γ a *solv manifold modelled on S* .

We require the following rigidity result of Mostow [10].

Theorem 2.1 (Mostow; [10]). *Let M_1 and M_2 be infrasolv manifolds. If $\pi_1(M_1) \cong \pi_1(M_2)$, then M_1 is diffeomorphic to M_2 .*

3 Cusps of Hilbert modular varieties

In this section, we prove Theorem 1.1. The philosophy for the proof is simple. Using the arithmeticity assumption on the torus bundle N , we construct an injective homomorphism $\rho: \pi_1(N) \longrightarrow \Delta(\mathrm{PSL}(2; \mathcal{O}_k))$. To find a Hilbert modular group Λ for which $\Delta(\Lambda) = \rho(\pi_1(N))$, we are reduced to making a subgroup separability argument. The proof is completed by applying Theorem 2.1. The remainder of this section is devoted to the details.

3.1 Subgroup separability

Recall that if G is a group, $H < G$ and $g \in G \setminus H$, we say H and g are *separated* if there exists a subgroup K of finite index in G which contains H but not g . We say that $H < G$ is *separable* in G if every $g \in G \setminus H$ and H can be separated.

As in [9], the main technical result we make use of is:

Theorem 3.1. *Let Λ be a Hilbert modular group and $\Delta(\Lambda)$, a maximal peripheral subgroup. Then every subgroup of $\Delta(\Lambda)$ is separable in Λ .*

3.2 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1. The following establishes a correspondence between k -arithmetic torus bundle groups and maximal peripheral subgroups of Hilbert modular groups.

Theorem 3.2 (Correspondence theorem). *Let N be a k -arithmetic torus bundle. Then there exists a faithful representation $\psi: \pi_1(N) \longrightarrow \Delta(\mathrm{PSL}(2; \mathcal{O}_k))$ such that $\psi(\pi_1(N))$ is a finite index subgroup of $\Delta(\mathrm{PSL}(2; \mathcal{O}_k))$. Moreover, there exists a finite index subgroup Λ of $\mathrm{PSL}(2; \mathcal{O}_k)$ such that $\Delta(\Lambda) = \psi(\pi_1(N))$.*

We defer the proof of Theorem 3.2 for the moment in order to prove Theorem 1.1.

Proof (Proof of Theorem 1.1). For the direct implication, since N is diffeomorphic to a cusp cross-section of a Hilbert modular variety, there exists a Hilbert modular group Λ and an isomorphism $\psi: \pi_1(N) \longrightarrow \Delta(\Lambda)$. To obtain an injective homomorphism $\rho: \pi_1(N) \longrightarrow k \rtimes k_+^\times$ such that $\rho(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$, we argue as follows. By conjugating by an element γ of $\mathrm{PSL}(2; k)$, we can assume that

$$\gamma^{-1} \psi(\pi_1(N)) \gamma \subset B_k = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} : \alpha \in k, \beta \in k_+^\times \right\}.$$

As $\gamma \in \mathrm{PSL}(2; k)$, $\gamma^{-1} \Lambda \gamma$ remains a Hilbert modular group, and moreover, $\gamma^{-1} \psi(\pi_1(N)) \gamma$ is commensurable with

$$\Delta(\mathrm{PSL}(2; \mathcal{O}_k)) = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} : \alpha \in \mathcal{O}_k, \beta \in \mathcal{O}_{k,+}^\times \right\}.$$

To obtain the faithful representation ρ , we simply compose $\mu_\gamma \circ \psi$ with the isomorphism $\iota: B_k \longrightarrow k \rtimes k_+^\times$ given by $\iota \left(\begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \right) = (\alpha, \beta)$.

For the reverse implication, we apply Theorem 3.2 and Theorem 2.1. Specifically, let Λ be the Hilbert modular group guaranteed by Theorem 3.2 and let N' denote an embedded cusp cross-section associated with $\Delta(\Lambda)$. As a smooth manifold, N' is of the form $\mathbf{R}^{2n-1} / \Delta(\Lambda)$. By Theorem 3.2, we have an isomorphism $\psi: \pi_1(N) \longrightarrow \pi_1(N')$. Applying Theorem 2.1, we obtain the desired diffeomorphism between N and N' .

In the proof of Theorem 3.2, the following lemma is required.

Lemma 3.3. *Let N be a k -arithmetic torus bundle. Then there exists an injective homomorphism $\rho: \pi_1(N) \longrightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Moreover, $\rho(\pi_1(N))$ is a finite index subgroup of $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$.*

Proof. Since N is k -arithmetic, we have a faithful representation $\theta: \pi_1(N) \longrightarrow k \rtimes k_+^\times$ such that $\theta(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Hence, given $(\alpha, \beta) \in \theta(\pi_1(N))$, we have for some $m \in \mathbf{N}$,

$$(\alpha + \beta\alpha + \beta^2\alpha + \cdots + \beta^{m-1}\alpha, \beta^m) \in \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times.$$

Consequently, $\beta^m \in \mathcal{O}_{k,+}^\times$ and thus $\beta \in \mathcal{O}_{k,+}^\times$. Even so, it may be the case that (α, β) is not contained in $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. This is rectified as follows. Select a generating set for $\pi_1(N)$, say g_1, \dots, g_u . For each generator, we have $\theta(g_j) = (\alpha_j, \beta_j)$ with $\alpha_j \in k$ and $\beta_j \in \mathcal{O}_{k,+}^\times$. Since k is the field of fractions of \mathcal{O}_k , we can select $\lambda_j \in \mathcal{O}_k$ such that $(0, \lambda_j)\theta(g_j)(0, \lambda_j)^{-1} \in \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Note that

$$(0, \lambda_j)\theta(g_j)(0, \lambda_j)^{-1} = (\lambda_j\alpha_j, \beta_j),$$

and so the second coordinate β_j is unchanged. Finally, for $\lambda = \lambda_1 \dots \lambda_u$, define $\rho = \mu_{(0, \lambda)} \circ \theta$, where $\mu_{(0, \lambda)}$ denotes the inner automorphism determined by $(0, \lambda)$. By construction, ρ is a faithful representation of $\pi_1(N)$ onto a finite index subgroup of $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$.

With Lemma 3.3 in hand, we prove Theorem 3.2.

Proof (Proof of Theorem 3.2). By Lemma 3.3, we have an injective homomorphism $\rho: \pi_1(N) \longrightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$ such that $\rho(\pi_1(N))$ is a finite index subgroup. To obtain the injective homomorphism ψ , we compose ρ with the isomorphism

$$\iota^{-1}: \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times \longrightarrow \Delta(\mathrm{PSL}(2; \mathcal{O}_k))$$

where $\iota^{-1}(\alpha, \beta) = \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix}$. That ψ is faithful and $\psi(\pi_1(N))$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))$ follow immediately from the properties of ρ and ι .

To find the desired subgroup Λ , we apply Theorem 3.1. Specifically, select a complete set of coset representatives $\gamma_1, \dots, \gamma_s$ for $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))/\psi(\pi_1(N))$. By Theorem 3.1, $\psi(\pi_1(N))$ is separable. Therefore for each j we can find finite index subgroups Λ_j such that $\gamma_j \notin \Lambda_j$ and $\psi(\pi_1(N)) < \Lambda_j$. To get the desired Λ , take $\Lambda = \bigcap_{j=1}^s \Lambda_j$.

3.3 A question of Hirzebruch

Let k be a totally real number field, $M < k$ an additive group of rank n (the degree of k over \mathbf{Q}), and $V < \mathcal{O}_{k,+}^\times$ a finite index subgroup such that for all $\lambda \in V$, $\lambda M \subset M$. For each pair (M, V) , we define the peripheral group

$$\triangle(M, V) = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} : \alpha \in M, \beta \in V \right\} < \mathrm{PSL}(2; k).$$

For any Hilbert modular variety, the peripheral groups $\triangle(\Lambda)$ are conjugate (in $\mathrm{PSL}(2; k)$) to groups of the form $\triangle(M, V)$. In [6, p. 203], Hirzebruch mentions that it is apparently unknown whether or not every $\triangle(M, V)$ can occur as a maximal peripheral subgroup of a Hilbert modular group. The following corollary gives an affirmative answer.

Corollary 3.4. *For every pair (M, V) , there exists a Hilbert modular group Λ such that $\triangle(\Lambda) = \triangle(M, V)$.*

Proof. As in the proof of Lemma 3.3, we can conjugate $\triangle(M, V)$ by an element of the form $\gamma = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, with $\lambda \in \mathcal{O}_k$, such that $\gamma^{-1} \triangle(M, V) \gamma$ is contained in $\mathrm{PSL}(2; \mathcal{O}_k)$. Since M and V are finite index subgroups of \mathcal{O}_k and $\mathcal{O}_{k,+}^\times$, respectively, $\gamma^{-1} \triangle(M, V) \gamma$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2; \mathcal{O}_k))$. Thus there exists a finite index subgroup $\Lambda_1 < \mathrm{PSL}(2; \mathcal{O}_k)$ such that $\triangle(\Lambda_1) = \gamma^{-1} \triangle(M, V) \gamma$. Hence, for $\Lambda = \gamma \Lambda_1 \gamma^{-1}$, we have $\triangle(\Lambda) = \triangle(M, V)$. As $\gamma \in \mathrm{PSL}(2; k)$, Λ is a Hilbert modular group, as required.

4 A simple criterion for arithmeticity

In this section, we give a simple criterion for the arithmeticity of (n, m) -torus bundles. The need for such a result is practical, as it allows one to establish the arithmeticity of a torus bundle computationally. We encourage the reader to compare the results of this section with Corollary 5.5 in [9].

4.1 Linear equations and presentations of torus bundle groups

For an (orientable) $(n, n-1)$ -torus bundle M , since both the base and fiber are aspherical, we have the short exact sequence induced by the long exact sequence of the fiber bundle

$$1 \longrightarrow \mathbf{Z}^n \longrightarrow \pi_1(M) \longrightarrow \mathbf{Z}^{n-1} \longrightarrow 1.$$

The action of \mathbf{Z}^{n-1} on \mathbf{Z}^n induces a homomorphism $\varphi: \mathbf{Z}^{n-1} \longrightarrow \mathrm{SL}(n; \mathbf{Z})$ called the *holonomy representation*. Since peripheral subgroups in Hilbert modular groups have faithful holonomy representation, we assume throughout that φ is faithful. In particular, we obtain a faithful representation of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n; \mathbf{Z})$.

Of primary importance for us here is that the holonomy representation together with any finite presentation yields a homogenous linear system of equations with coefficients in \mathbf{Z} . This system arises as follows. For ease, select a presentation of the form

$$\langle x_1, \dots, x_n, \overline{y_1}, \dots, \overline{y_{n-1}} : R \rangle$$

where x_1, \dots, x_n generate \mathbf{Z}^n , $\overline{y_1}, \dots, \overline{y_{n-1}}$ are lifts of a generating set y_1, \dots, y_{n-1} for \mathbf{Z}^{n-1} , and R is a finite set of relations of the form

$$x_j \overline{y_k} = \overline{y_k} w_{j,k}, \quad w_{j,k} \in \langle x_1, \dots, x_n \rangle.$$

Using the holonomy representation, we can write

$$x_j = (a_j, I), \quad \overline{y_j} = (b_j, \varphi(y_j)) \in \mathbf{R}^n \rtimes \mathrm{SL}(n; \mathbf{R}).$$

Each relation in the presentation yields a linear homogenous equation in the vector variables a_j and b_j (see below for an explicit example of how these equations arise). Namely, we insert the above forms for x_j and $\overline{y_k}$ into the relation and consider only the first coordinate. The equations we obtain are of the form

$$a_j + b_k - \varphi(y_k) - v_{j,k} = 0$$

where $w_{j,k} = (v_{j,k}, I)$. That this system has integral solutions which yield faithful representations follows from the fact that φ is faithful and induces a faithful representation of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n; \mathbf{Z})$.

4.2 A simple criterion for arithmeticity

The main result of this section is a simple criterion for arithmeticity based on the structure of the holonomy representation. In the statement and proof, let $\mathrm{Res}_{k/\mathbf{Q}}$ denotes restriction of scalars from k to \mathbf{Q} and assume that $[k : \mathbf{Q}] = n$ and $\mathrm{rank} \mathcal{O}_k^\times = n - 1$. In particular, k is totally real.

Theorem 4.1. *Let M be an orientable $(n, n-1)$ -torus bundle. Then M is diffeomorphic to a cusp cross-section of a Hilbert modular variety defined over k if and only if $\varphi = \mathrm{Res}_{k/\mathbf{Q}}(\chi)$, for some faithful character $\chi: \mathbf{Z}^{n-1} \longrightarrow \mathcal{O}_{k,+}^\times$, where φ is some holonomy representation.*

Proof. For the direct implication, since M is diffeomorphic to a cusp cross-section of a Hilbert modular variety, by Theorem 1.1, we have a faithful representation

$$\rho: \pi_1(M) \longrightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$$

By restricting scalars from k to \mathbf{Q} , we obtain a faithful representation

$$\mathrm{Res}_{k/\mathbf{Q}}(\rho): \pi_1(M) \longrightarrow \mathbf{Z}^n \rtimes \mathrm{SL}(n; \mathbf{Z}).$$

The proof is completed by noting that the holonomy map induced by this representation is simply $\text{Res}_{k/\mathbf{Q}}(\chi)$, where $\chi: \mathbf{Z}^{n-1} \rightarrow \mathcal{O}_{k,+}^\times$ is the holonomy representation induced by the representation ρ .

For the converse, we seek a faithful representation $\rho: \pi_1(M) \rightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. Note that since $[k: \mathbf{Q}] = n$ and $\text{rank } \mathcal{O}_k^\times = n-1$, the image of $\pi_1(M)$ would necessarily be a finite index subgroup. By assumption, we have a faithful character $\chi: \mathbf{Z}^{n-1} \rightarrow \mathcal{O}_{k,+}^\times$. We extend this to a faithful representation of $\pi_1(M)$ into $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$ as follows. Select a presentation as above for $\pi_1(M)$ with generators $x_1, \dots, x_n, \overline{y_1}, \dots, \overline{y_{n-1}}$. Write

$$x_i = (\alpha_i, 1), \overline{y_i} = (\gamma_i, \chi(y_i)) \in k \rtimes \mathcal{O}_{k,+}^\times \quad (1)$$

where α_i and γ_i are to be determined. Using our presentation for $\pi_1(M)$, we obtain a system of linear homogenous equations \mathcal{L} with coefficients in \mathcal{O}_k . Note, as above, solutions to \mathcal{L} yield representations of $\pi_1(M)$ into $k \rtimes \mathcal{O}_{k,+}^\times$. We assert that there is a solution which yields a faithful representation. To see this, by restricting scalars from k to \mathbf{Q} , we obtain a linear system $\text{Res}_{k/\mathbf{Q}}(\mathcal{L})$ with coefficients in \mathbf{Z} . Solutions to the system $\text{Res}_{k/\mathbf{Q}}(\mathcal{L})$ yield representations of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \text{SL}(n; \mathbf{Z})$. Moreover, a solution to $\text{Res}_{k/\mathbf{Q}}(\mathcal{L})$ which yields a faithful representation is equivalent to a solution of \mathcal{L} which yields a faithful representation into $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$. That such a solution exists with integral coefficients for $\text{Res}_{k/\mathbf{Q}}(\mathcal{L})$ follows from the faithfulness of $\text{Res}_{k/\mathbf{Q}}(\chi)$ and our discussion in the previous subsection. This yields a solution for \mathcal{L} with coefficients in \mathcal{O}_k which yields a faithful representation. Therefore, M is k -arithmetic, since there exists a faithful representation $\psi: \pi_1(M) \rightarrow \mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$ such that $\psi(\pi_1(M))$ is a finite index subgroup of $\mathcal{O}_k \rtimes \mathcal{O}_k^\times$.

Remark. If the character χ only maps into \mathcal{O}_k^\times , the above proof yields a faithful representation $\rho: \pi_1(M) \rightarrow \mathcal{O}_k \rtimes \mathcal{O}_k^\times$.

5 Sol 3-manifolds

Before proving Theorem 1.2, we give a brief review of Sol 3-manifolds (see [14]). Let $\text{Sol} = \mathbf{R}^2 \times \mathbf{R}^+$ with group operation defined by

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) \stackrel{\text{def}}{=} (x_1 + e^{t_1} x_2, y_1 + e^{-t_1} y_2, t_1 + t_2).$$

By a Sol 3-orbifold, we mean a manifold M which is diffeomorphic to Sol/Γ , where Γ is a discrete subgroup of $\text{Aff}(\text{Sol})$ such that Sol/Γ is compact and $[\Gamma: \Gamma \cap \text{Sol}] < \infty$. These manifolds, in the terminology from §2, are infrasolv manifolds modelled on Sol. However, the terminology used in this section for these manifolds is more prevalent.

In [14], Scott proved that every $(2, 1)$ -torus bundles admits either a Euclidean, Nil, or Sol structure. The following result is easily derived from [14]. We include a proof here for completeness.

Proposition 5.1. *Let M be an orientable $(2, 1)$ -torus bundle which admits a Sol structure. Then there exists a faithful representation $\rho: \pi_1(M) \rightarrow \mathcal{O}_k \rtimes \mathcal{O}_k^\times$ for some real quadratic number field k .*

Proof. For any $(2, 1)$ -torus bundle M , let the \mathbf{Z} -action be given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If the order of A is finite, then $\pi_1(M)$ is a Bieberbach group and M admits a Euclidean structure. Therefore we may assume that the order of A is infinite. If A is not diagonalizable, then some power of A is conjugate to $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ with $\alpha \neq 0$. In this case, M admits a Nil structure. Thus, we may assume that A is diagonalizable. In this case we have $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ for a conjugate of A . It follows, since $A \in \mathrm{SL}(2; \mathbf{Z})$, that β and β^{-1} are algebraic integers in the real quadratic field $\mathbf{Q}(\beta)$. Thus the representation $\varphi: \mathbf{Z} \rightarrow \mathrm{GL}(2; \mathbf{Z})$ is conjugate to $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$, where $\chi: \mathbf{Z} \rightarrow \mathcal{O}_k^\times$ is given by $\chi(1) = \beta$. Therefore by the remark following Theorem 4.1, we have a faithful representation $\rho: \pi_1(M) \rightarrow \mathcal{O}_k \rtimes \mathcal{O}_k^\times$, as asserted.

Via Proposition 5.1, note every Sol 3-manifold group does faithfully represent into $\mathrm{Isom}((\mathbf{H}^2)^2)$. Those that arise as cusp cross-sections of Hilbert modular surfaces are precisely the ones whose fundamental group faithfully represents into the identity component of $\mathrm{Isom}((\mathbf{H}^2)^2)$. However, the quotients of those groups which fail to map into the identity component do produce finite volume quotients which possess 2-fold covers which are Hilbert modular surfaces. For this reason, we call such quotients *generalized Hilbert modular varieties*. Given this, Theorem 1.2 follows from this discussion in combination with Theorem 1.1.

6 Geometric bounding

Let W be a 1-cusped Hilbert modular surface W with torsion free fundamental group—we call W a *Hilbert modular manifold* in this case. Similar to the thick-thin decomposition of a real hyperbolic n -manifold, W has a decomposition comprised of a compact manifold \tilde{W} with boundary S and cusp end $S \times \mathbf{R}^+$. Following Schwartz [13] (see also [4]), we call the universal cover of \tilde{W} the associated *neutered manifold*, and note \tilde{W} is a compact 4-manifold with Sol 3-manifold boundary. Moreover, the locally symmetric metric \tilde{g} on W restricted to S endows S with a complete Sol metric g such that \tilde{g} is a complete, finite volume metric in the interior of \tilde{W} and (S, g) is a totally geodesic boundary.

The goal of this section is the establishment of a nontrivial obstruction for this geometric situation. The obstruction is obtained by mimicking the argument of Long–Reid [7] for flat 3-manifolds. This in combination with a calculation of Hirzebruch bears Theorem 1.3 from the introduction.

In [6], Hirzebruch extended his signature formula to Hilbert modular surfaces. The formula relates the signature of the neutered manifold \tilde{W} to a Hirzebruch L -polynomial evaluated on the Pontrjagin classes of \tilde{W} but with a correction term associated to $\partial\tilde{W}$. When $\pi_1(W)$ contains torsion, the elliptic singularities also contribute nontrivially to this correction term, and so for simplicity, we assume throughout that $\pi_1(W)$ is torsion free. In this case, Hirzebruch’s formula becomes

$$\sigma(\tilde{W}) = \delta(E_1) + \cdots + \delta(E_r)$$

where E_1, \dots, E_r is a complete set of cusp ends of W given from the thick-thin decomposition and $\sigma(\tilde{W})$ denotes the signature of \tilde{W} . The definition of the terms $\delta(E_j)$ are given as follows. Associated to each cusp end is the $\pi_1(W)$ -conjugacy class of a maximal peripheral subgroup Γ_j . The group Γ_j is conjugate in $\mathrm{PSL}(2; k)$ to a subgroup of the familiar form $\triangle(M_j, V_j)$. In turn, for the pair (M_j, V_j) , we have an associated Shimizu L -function $L(M_j, V_j, s)$ —see [15]—defined by

$$L(M, V, s) = \sum_{\beta \in (M_j \setminus \{0\})/V_j} \frac{\mathrm{sign}(N_{k/\mathbf{Q}}(\beta))}{(N_{k/\mathbf{Q}}(\beta))^s}$$

where $N_{k/\mathbf{Q}}$ is the norm map. With this, the invariant $\delta(E_j)$ is defined to be

$$\delta(E_j) = \frac{-\mathrm{vol}(M_j)}{\pi^2} L(M_j, V_j, 1)$$

where $\mathrm{vol}(M_j)$ is the volume of \mathbf{R}^2/M with respect to the pairing $\mathrm{Tr}_{k/\mathbf{Q}}$. Equivalently,

$$\mathrm{vol}(M_j) = \left| \det(\beta_i^{(j)}) \right|,$$

where β_1, β_2 is a \mathbf{Z} -module basis for M_j and $\beta_i^{(1)}$ and $\beta_i^{(2)}$ denote the image of β_i under the two real embeddings of k into \mathbf{R} .

Theorem 6.1 (Hirzebruch; [6]). *If W is a Hilbert modular manifold with exactly one cusp, then*

$$\sigma(\tilde{W}) = \frac{-\mathrm{vol}(M)}{\pi^2} L(M, V, 1)$$

for the unique $\pi_1(W)$ -conjugacy class $\triangle(M, V)$.

As we seek an integrality condition, it is convenient to change the pair M, V . Associated to the \mathbf{Z} -module M is the *dual lattice* M^* defined to be the image of M under the duality pairing provided by $\mathrm{Tr}_{k/\mathbf{Q}}$.

Proposition 6.2. *For a horosphere \mathcal{H} stabilized by $\triangle(M, V)$ and $\triangle(M^*, V)$, $\mathcal{H}/\triangle(M, V)$ and $\mathcal{H}/\triangle(M^*, V)$ are diffeomorphic Sol 3-manifolds.*

Proof. Let $\varphi_M, \varphi_{M^*}: V \longrightarrow \mathrm{SL}(2; \mathbf{Z})$ be the holonomy representations for $\triangle(M, V)$ and $\triangle(M^*, V)$. The pairing $\mathrm{Tr}_{k/\mathbf{Q}}$ can be viewed as an element of $\lambda \in \mathrm{SL}(2; \mathbf{Z})$ such that $\lambda M = M^*$. By construction $\varphi_{M^*} = \lambda(\varphi_M)\lambda^{-1}$, and so we have an isomorphism $\rho: \triangle(M, V) \longrightarrow \triangle(M^*, V)$ given by

$$\rho(\beta, \varphi_M(\alpha)) = (\lambda\beta, \lambda\varphi_M(\alpha)\lambda^{-1}).$$

The proof is completed by appealing to the smooth rigidity theorem of Mostow Theorem 2.1.

Hecke (see [1]) related the L -functions $L(M, V, s)$ and $L(M^*, V, s)$ by the functional equation $H(M, V, s) = (-1)^s H(M^*, V, 1-s)$, where

$$H(M, V, s) = \left[\Gamma\left(\frac{s+1}{2}\right) \right]^2 \pi^{-(s+1)} [\mathrm{vol}(M)]^s L(M, V, s)$$

The specialization of this functional equation at $s = 1$ produces

$$\begin{aligned} (\Gamma(1))^2 \pi^{-2} \text{vol}(M) L(M, V, 1) &= - \left(\Gamma\left(\frac{1}{2}\right) \right)^2 \pi^{-1} L(M^*, V, 0) \\ L(M^*, V, 0) &= - \frac{\text{vol}(M)}{\pi^2} L(M, V, 1), \end{aligned}$$

and thus from this and Theorem 6.1, we obtain

$$\sigma(\tilde{W}) = L(M^*, V, 0). \quad (2)$$

It is at this point that we take stock in what has been done. For a 1-cusped Hilbert modular manifold W with cusp cross-section S , we have associated to S the invariant $\delta(S \times \mathbf{R}^+)$. As both M and V depend on the associated Sol metric on S afforded by its embedding as a cusp cross-section, the invariant $\delta(S \times \mathbf{R}^+)$ depends on the associated Sol metric on S . Our goal is to use the integrality of $\sigma(\tilde{W})$ and (2) to produce an obstruction for S to topologically occur in this geometric setting. For this, it remains to show the invariant $\delta(S \times \mathbf{R}^+)$ is independent of the Sol structure on S .

Given a peripheral group $\Delta(M, V)$ and stabilized horosphere \mathcal{H} , the metric on $\mathbf{H}_{\mathbf{R}}^2 \times \mathbf{H}_{\mathbf{R}}^2$ endows \mathcal{H} with a $\Delta(M, V)$ -invariant metric $g_{\mathcal{H}, M, V}$. Consequently the metric $g_{\mathcal{H}, M, V}$ descends to quotient $\mathcal{H}/\Delta(M, V)$ and endows $\mathcal{H}/\Delta(M, V)$ with a complete Sol structure that depends on the horosphere \mathcal{H} only up to similarity.

The formula (2) was also established in [1] where $L(M^*, V, 0)$ was reinterpreted as the η -invariant of an adiabatic limit.

Theorem 6.3 (Atiyah–Donnelly–Singer;[1]).

$$L(M^*, V, 0) = \lim_{\varepsilon \rightarrow 0} \eta(\mathcal{H}/\Delta(M^*, V), g_{\mathcal{H}, M^*, V}/\varepsilon).$$

More generally, given any Sol structure g on S , we can define

$$\delta(S, g) = \lim_{\varepsilon \rightarrow 0} \eta(S, g/\varepsilon).$$

The last ingredient for proof of Theorem 1.3 is the independence of $\delta(S, g)$ from g , a result established by Cheeger and Gromov [3].

Theorem 6.4 (Cheeger–Gromov;[3]). *$\delta(S, g)$ is a topological invariant of the Sol 3-manifold S .*

We are now in position to state and prove the principal observation needed in the proof of Theorem 1.3 (compare with [7]).

Theorem 6.5. *If S is diffeomorphic to a cusp cross-section of a 1-cusped Hilbert modular manifold, then $\delta(S) \in \mathbf{Z}$.*

Proof. If (S, g) arises as a cusp cross-section of a 1-cusped Hilbert modular manifold W , then there is an isometric embedding $f: (S, g) \rightarrow W$ onto a cusp cross-section of W . Let $f_*(\pi_1(S)) = \Delta(M, V)$ with associated horosphere \mathcal{H} selected such that

$\mathcal{H}/\Delta(M, V)$ is embedded in W . By Proposition 6.2, $\mathcal{H}/\Delta(M^*, V)$ is diffeomorphic to S , though equipped with the metric $g_{\mathcal{H}, M^*, V}$. From the computation above in combination with Theorem 6.3, $\sigma(\tilde{W}) = \delta(S, g_{\mathcal{H}, M^*, V})$ and by Theorem 6.4, the right hand side depends only on the topological type of S . Since $\sigma(\tilde{W})$ is in \mathbf{Z} , $\delta(S)$ is in \mathbf{Z} as asserted.

Proof (Proof of Theorem 1.3). To prove Theorem 1.3, by Theorem 6.5, it suffices to find a Sol 3-manifold S for which $\delta(S) \notin \mathbf{Z}$. For $k = \mathbf{Q}(\sqrt{3})$, the standard Hilbert modular surface W over k has precisely one cusp, since the number of cusps of a standard Hilbert modular surface over k is the ideal class number of k . Setting S to be an embedding cusp cross-section of W , the proof is completed by appealing to [6]. Specifically, Hirzebruch showed $\delta(S) = -1/3$.

Remark. It is unknown to the author whether or not there exist 1-cusped Hilbert modular manifolds. In addition, the number fields $\mathbf{Q}(\sqrt{6})$, $\mathbf{Q}(\sqrt{21})$ and $\mathbf{Q}(\sqrt{33})$ also have standard Hilbert modular surfaces with precisely one cusp for which the associated invariant $\delta(S) \notin \mathbf{Z}$. In each of these cases, $\delta(S) = -2/3$ (see [6, p. 236]).

It is unknown to the author whether or not there exist 1-cusped Hilbert modular manifolds. Using the generalized Riemann hypothesis, K. Petersen [11] constructed infinite many 1-cusped Hilbert modular surfaces. However, the nature of the construction likely produces Hilbert modular surface groups with 2-torsion.

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